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Note on the pure bending of thin non-isotropic plates in the presence of couple stresses.

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Effects of couple stresses on the pure bending of plates in some simple cases of isotropic material have recently been discussed by Koiter (1964) and Hoffman (1964). The object of of this note is to prove that simple expressions for the displacements and bending moments can also be obtained when the material of the plate has either orthogonal elastic symmetry or is cylindrically aeolotropic.

Introduction

In a Cosserat continuum we have not only the small strain tensor

$$\Sigma_{ij} = \tfrac{1}{2}(u_{ij} + u_{ji}) \text{ (cf. Koiter 1964, equation 31)} \qquad (1)$$

but also the twist curvature tensor

$$k_{ij} = \omega_{ji} \qquad (2)$$

corresponding to the deformation. Components of K_{ij} with $i=j$ are twists and those with $i \neq j$ are curvatures. u_i is the displacement vector and,

$$\omega_i = \tfrac{1}{2} e_{ijk}\, u_{kj}$$

is the small rotation vector. The couple stress tensor μ_{ij} is given by the equations

$$\mu_{ij} = B_1 k_{ij} + B_2 k_{ji}$$

where B_1 and B_2 are moduli of curvatures.

Equations of equilibrium in the absence of body forces and body couples reduce to

$$\sigma_{jkj} = 0. \quad ..(3) \quad \text{and} \quad \mu_{jij} = l_{ijk}\, \sigma_{jk} = 0 \quad ...(4)$$

Material with orthogonal symmetry

Solution of the problem : The stress strain relations in the case of material with three orthogonal planes of elastic symmetry are given by (cf. Love p. 161)

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \end{pmatrix} = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix} \begin{pmatrix} l_{xx} \\ l_{yy} \\ l_{zz} \end{pmatrix} \qquad ...(5)$$

$$\begin{pmatrix} \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} L \\ M \\ N \end{pmatrix} \begin{pmatrix} l_{yx} \\ l_{zx} \\ l_{xy} \end{pmatrix} \qquad \ldots(6)$$

We assume the displacements to be given by

$$\left.\begin{aligned} u &= k_1 zx \\ v &= k_2 yz \\ w &= -\frac{k_1}{2}x^2 + \frac{k_2}{2}y^2 + \frac{P}{2}z^2 \end{aligned}\right\} \qquad \ldots(7)$$

where k_1 and k_2 are components of curvatures and P is a constant to be determined. By equation (1) the non-zero stress components are $\epsilon_{11} = k_1 z$, $\epsilon_{22} = -k_2 z$, $\epsilon_{33} = Pz$ From equation (2) the non-zero twist curvature components are obtained as $k_{12} = k_1$, $k_{21} = k_2$. (8)

Values assumed for the displacements make

$$\tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0\,;$$

τ_{zz} will also be zero if

$$CP = Fk_2 - Gk_1 \qquad \ldots(9)$$

It is also found that the stress equations of equilibrium are satisfied. The statistically equivalent stress couples per unit width of section for a plate of thickness h are

$$\begin{aligned} M_{xy} &= \int_{-h/2}^{h/2} (z\,\tau_{xx} + \mu_{xy})\,dz \\ &= (Ak_1 - Hk_2 + GP)\,\frac{h^3}{12} + h\,(B_1k_1 + B_2k_2) \end{aligned} \qquad \ldots(10)$$

$$\begin{aligned} M_{yx} &= \int_{-h/2}^{h/2} (-z\,\tau_{yy} + \mu_{yx})dz \\ &= -(Hk_1 - Bk_2 + FP)\,\frac{h^3}{12}\,(B_1k_2 + B_2k_1). \end{aligned} \qquad \ldots(11)$$

where B_1 and B_2 are the moduli of curvatures characteristic of the material. The Curvatures in terms of the stress couples are given by

$$k_1 = \frac{\left(M_{xy} - \frac{PGh^3}{12}\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Hh^3}{11}\right)\left(M_{yx} + \frac{PFh^3}{12}\right)}{\left(\frac{Ah^3}{12} + hB_1\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2} \qquad \ldots(12)$$

$$k_2 = \frac{\left(M_{yx} + \frac{PFh^3}{12}\right)\left(\frac{Ah^3}{1h} + hB_1\right) - \left(hB_2 - \frac{Hh^3}{12}\right)\left(M_{xy} - \frac{GPh^3}{12}\right)}{\left(\frac{Ah^3}{12} + hB_1\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2} \quad \ldots(13)$$

Special case (I) : Uniaxial bending.

With $M_{yx} = 0$

$$k_1 = \frac{\left(M_{xy} - \frac{PGh^3}{12}\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \frac{PFh^3}{12}\left(hB_2 - \frac{Hh^3}{12}\right)}{\left(\frac{Ah^3}{12} + hB_1\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2}$$

$$k_2 = \frac{\frac{PFh^3}{12}\left(\frac{Ah^3}{12} + B_1 h\right) - \left(hB_2 - \frac{Hh^3}{12}\right)\left(M_{xy} - \frac{GPh^3}{12}\right)}{\left(\frac{Ah^3}{12} + B_1 h\right)\left(B_1 h + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2}.$$

Special case (II) : Cylindrical bending.

The moment curvature relations corresponding to those derived by Koiter and Hoffman may be obtained by putting $k_2 = 0$

$$M_{xy} = (Ak_1 + GP)\frac{h^3}{12} + hB_1 k_1$$

$$M_{yx} = k_1\left(hB_2 - \frac{Hh^3}{12}\right)\frac{PFh^3}{12}$$

Special case : (III) Pure twist.

when $M_{xy} = M_{yx}$,

$$k_2 = \frac{M_{xy}\left[h(B_1 - B_2) + \frac{h^3}{12}(B + H)\right] - \frac{Ph^4}{12}(GB_1 + FB_2) - \frac{Ph^6}{144}(BG - FH)}{\left(\frac{Ah^3}{12} + hB_1\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2}$$

$$k_2 = \frac{M_{xy}\left[h(B_1 - B_2) + \frac{h^3}{12}(B + H)\right] + \frac{Ph^4}{12}(FB_1 + GB_2) + \frac{Ph^6}{144}(AF - GH)}{\left(\frac{Ah^3}{12} + hB_1\right)\left(hB_1 + \frac{Bh^3}{12}\right) - \left(hB_2 - \frac{Bh^3}{12}\right)^2}$$

Cylindrical Aeolotropy

In the of case of cylindrical aeolotropy we have the stress strain relations

$$\begin{pmatrix} \tau_{rr} \\ \tau_{\theta\theta} \\ \tau_{zz} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{11} & c_{13} \\ c_{13} & c_{13} & c_{33} \end{pmatrix} \begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{bmatrix}, \qquad c_{ij} = c_{ji} \quad \ldots(1)$$

$$\tau_{\theta z} = c_{44}\, e_{\theta z}\; \tau_{z r} = c_{44}\, e_{z r},\; \tau_{r\theta} = c_{66}\, e_{r\theta} \quad \dots (2)$$

For the uniform bending of a circular plate we assume in this case

$$\left.\begin{aligned} u_r &= k_{rz} \\ u_\theta &= 0 \\ u_z &= -\frac{k}{2} r^2 + \frac{P}{2}\, z^2 \end{aligned}\right\} \quad \dots (3)$$

where P is a constant to be determined Here (r, θ, z) are the cylindrical coordinates with the origin at the centre of the plate and the axis of z perpendicular to it.

With the above components of displacement the stress components become

$$\left.\begin{aligned} \tau_{rr} &= z\,(kc_{11} + kc_{12} + Pc_{13}). \\ \tau_{\theta\theta} &= z\,(kc_{12} + kc_{11} + Pc_{13}). \\ \tau_{zz} &= z\,(kc_{13} + kc_{13} + Pc_{33}). \end{aligned}\right\} \quad \dots (4)$$

$$\tau_{\theta z} = \tau_{z r} = \tau_{r\theta} = 0. \quad \dots (5)$$

It is found that the stress equations of equilibrium are satisfied.

The rotation component $\omega_\theta = \dfrac{1}{2}\left(\dfrac{\partial ur}{\partial z} - \dfrac{\partial uz}{\partial r}\right) = k_r$ $\quad \dots (6)$

and the couple stress component μ_r is given by

$$\mu_r = B\,\frac{\partial}{\partial r}\,\omega_\theta = B_k \quad \dots (7)$$

The statically equivalent stress couple M_r is given by

$$\begin{aligned} M_r &= \int_{-h/2}^{h/2} (z\tau_{rr} + \mu_r)\, dz \\ &= \frac{h^3}{12}\,(kc_{11} + kc_{12} + Pc_{13}) + Bkh. \end{aligned}$$

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